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A characterization of the interval distance monotone graphs ☆

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Abstract

A simple connected graph G is said to be interval distance monotone if the interval $I(u, v)$ between any pair of vertices u and v in G induces a distance monotone graph. Aïder and Aouchiche [Distance monotonicity and a new characterization of hypercubes, Discrete Math. 245 (2002) 55–62] proposed the following conjecture: a graph G is interval distance monotone if and only if each of its intervals is either isomorphic to a path or to a cycle or to a hypercube. In this paper we verify the conjecture.

Keywords: Hypercube; Interval; Distance monotone; Interval monotone; Interval distance monotone

1. Introduction

The intervals of graphs play a very interesting role in studying some classes of graphs that contain hypercubes as a subclass. Interval monotone graphs (every interval is convex) and distance monotone graphs (every interval is closed) were introduced by Mulder [9] and Burosch et al. [3,4], respectively. A new characterization of hypercubes was given by Burosch et al. [3]: a graph G with minimum degree at least three is a hypercube if and only if G is both distance monotone and interval monotone.

Recently Aïder and Aouchiche [1] introduced interval distance monotone graph: every interval induces a distance monotone graph. They also pointed out that hypercubes, geodesic graphs, extended odd graphs [9] and Hamming graphs [5] are interesting examples of interval distance monotone graphs; but there exists an interval distance monotone graph that is not distance monotone, interval monotone, nor weakly spherical (see next section for its definition). Two further characterizations of hypercubes were accordingly obtained: a graph G with minimum degree at least three is a hypercube if and only if G is both distance monotone and interval distance monotone, if and only if G is bipartite and interval distance monotone.

A question naturally arises: how to characterize the interval distance monotone graphs? Aïder and Aouchiche [1,2] proposed a conjecture: a graph G is interval distance monotone if and only if each of its intervals is either isomorphic to a path or to a cycle or to a hypercube. In this paper we show that this conjecture is correct (see Theorem 3.1).

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2. Preliminaries

All our graphs are finite, undirected and simple. Let $G = (V, E)$ be a connected graph and let S be a set of vertices of G. We denote by $G[S]$ the subgraph of G induced by S. Let u be a vertex of G. A vertex of G is called a *neighbour* of u if it is adjacent to u. Then the neighbourhood $N_G(u)$ of u consists of all neighbours of u. The degree of u, denoted by $d_G(u)$, is the number of neighbours of u. The maximum and the minimum degree in G will be denoted by $\Delta(G)$ and $\delta(G)$, respectively. A graph is called *r-regular* if all vertices have the same degree r.

The distance $d_G(u, v)$ between two vertices u and v of G is the length of a u, v-geodesic (i.e. a shortest path between u and v). The diameter of G, denoted as $D(G)$, is the maximum distance between any pairs of vertices. The *interval* $I_G(u, v)$ can be defined by

$$I_G(u, v) = \{w \in V | d_G(u, w) + d_G(w, v) = d_G(u, v)\},$$

that is, it consists of all vertices on u, v-geodesics. The length of $I_G(u, v)$ is defined to be the distance $d_G(u, v)$. For convenience, an interval sometimes means its induced subgraph. If no confusion arises, we will use the above notations without indicating the reference graph. For example, we will write $I(u, v)$ instead of $I_G(u, v)$.

Lemma 2.1 (*Burosch et al. [3]*). *For any $u, v \in V(G)$, we have that*

(i) $I_G(u, v) = I_G(v, u)$.
(ii) *If $x \in I_G(u, v)$, then $I_G(u, x) \subseteq I_G(u, v)$.*
(iii) *If $x \in I_G(u, v)$ and $y \in I_G(u, x)$, then $x \in I_G(y, v)$.*

Lemma 2.2. *Let $H := G[I_G(u, v)]$. Then the diameter of H is equal to the length of $I_G(u, v)$, i.e. $D(H) = d_G(u, v)$.*

Proof. It is obvious. □

The hypercube of dimension n, denoted by Q_n, is a graph, where the vertices are the n-tuples $b_1 b_2 \dots b_n$ with $b_i \in \{0, 1\}$, two vertices are adjacent if the corresponding tuples differ in precisely one place (note that $Q_0 = K_1$).

Lemma 2.3 (*Imrich and Klavžar [5]*). (i) *Q_n is connected, bipartite, n-regular and has diameter n;*
(ii) $|V(Q_n)| = 2^n$ *and* $|E(Q_n)| = n2^{n-1}$;
(iii) *for any pair of vertices $u, v \in V(Q_n)$, the subgraph induced by the interval $I_{Q_n}(u, v)$ is a hypercube of dimension $d(u, v)$.*

A subgraph H of a connected graph G is *convex* if, for any two vertices u and v of H, the interval $I_G(u, v)$ is contained in H [9]. A graph G is called *interval monotone*, if all its intervals are convex subgraphs of G. An interval $I(u, v)$ of G is *closed*, if for any vertex $w \in V \backslash I(u, v)$, there exists a vertex w' in $I(u, v)$ such that $d(w, w') > d(u, v)$. If all intervals of a graph G are closed, then G is said to be *distance monotone*.

Theorem 2.4 (*Burosch et al. [3]*). *Let G be a graph with minimum degree $\delta(G) \geqslant 3$. Then G is isomorphic to a hypercube if and only if G is both distance monotone and interval monotone.*

Aïder and Aouchiche [1] introduced the following graphs: a connected graph G is called *interval distance monotone* if for any two vertices u and v in G, the interval $I(u, v)$ induces a distance monotone graph.

Theorem 2.5 (*Aïder and Aouchiche [1]*). *Let G be a graph with minimum degree $\delta(G) \geqslant 3$. Then G is isomorphic to a hypercube if and only if G is bipartite and interval distance monotone.*

Assume that $w, \bar{w} \in S \subseteq V(G)$. If $d(w, \bar{w}) \geqslant d(u, v)$ for all u and v in S, $\bar{w}$ is a *diametrical* vertex of w in S. If any vertex of G has a unique diametrical vertex, then G is a *diametrical graph*. An interval $I(u, v)$ is weakly spherical if for any vertex w in $I(u, v)$, there exists a unique vertex $\bar{w}$ in $I(u, v)$ with $d(w, \bar{w}) = d(u, v)$. A graph G is said to be *weakly spherical*, if all its intervals are weakly spherical.

Theorem 2.6 (*Burosch et al. [3]*). *Let G be a distance monotone graph. Then*

(1) *G is bipartite.*
(2) *If v, w_1, w_2, w_3 are different vertices of G such that w_1, w_2, w_3 are adjacent to v, then there is a vertex u in G adjacent to w_1 and w_2 but not to w_3.*
(3) *If $\delta(G) = 1$, G is a path, and if $\delta(G) = 2$, G is a cycle of even length.*
(4) *If $\Delta(G) \geqslant 3$, then G is both weakly spherical and diametrical.*
(5) *$V(G) = I(u, v)$ for every $u, v \in V(G)$ such that $d(u, v) = D(G)$.*

3. Main theorem and its proof

Now we describe our main theorem which completely solves the conjecture on a characterization of interval distance monotone graphs, put forward by Aïder and Aouchiche.

Theorem 3.1. *A connected graph G is interval distance monotone if and only if each of its intervals is either isomorphic to a path or to an even cycle or to a hypercube.*

To prove the main theorem, we first establish some lemmas. The following one can be easily proved by Theorem 2.6(1) and (2) and was described in a recent paper [2].

Lemma 3.2 (*Aïder and Aouchiche [2]*). *Let G be an interval distance monotone graph. Then G contains neither $K_{1,1,2}$ nor $K_{2,3}$ as its induced subgraphs (see Figs. 1 and 2).*

The (0, 2)-graph was introduced by Mulder [8]: a connected graph is a (0, 2)-*graph* if any two distinct vertices have either exactly two neighbours in common or none at all.

Lemma 3.3. *Let G be an interval distance monotone graph. For any $u, v \in V(G)$, $I_G(u, v)$ is a (0, 2)-graph if its minimum degree is not less than 3.*

Proof. Let $I_G(u, v)$ be any interval of G and H its induced subgraph in G. Suppose that $\delta(H) \geqslant 3$. By Lemma 3.2, G and thus H contains no $K_{2,3}$ as its induced subgraph. Then any two distinct vertices in H have at most two neighbours

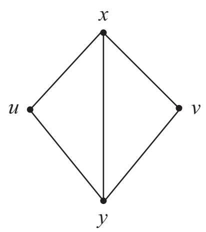

Fig. 1. $K_{1,1,2}$.

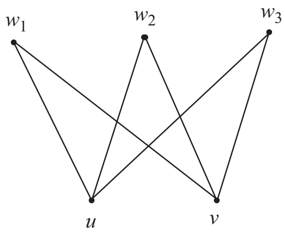

Fig. 2. $K_{2,3}$.

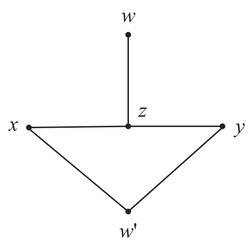

Fig. 3. Illustration for the proof of Lemma 3.3.

in common. Otherwise two vertices x and y have three neighbours w_1, w_2 and w_3 in common. Since H is a distance monotone graph, by Theorem 2.6(1) H is bipartite. Then x and y are non-adjacent, and no two of w_1, w_2 and w_3 are adjacent. Hence H contains a $K_{2,3}$ as its induced subgraph, a contradiction. Suppose that two vertices x and y of H have exactly one neighbour z in common (see Fig. 3). Since $d_H(z) \geqslant 3$, we can choose another neighbour w of z, other than x, y. By Theorem 2.6(2) there must exist a vertex $w' \in V(H)$, which is adjacent to x and y but not to w. Then w' is another common neighbour of x and y, which contradicts the previous supposition. Hence H is a (0, 2)-graph. $\square$

Let G be a connected graph with a specified vertex u. Let $N_k := \{w \in V(G) : d_G(u, w) = k\}$, $k = 0, 1, \ldots$. Then $N_i \cap N_j = \emptyset$ for any $i \neq j$. The finite sequence $N_0, N_1, \ldots$ is called the *layer decomposition* of G from the vertex u.

Lemma 3.4. *Let $N_0, N_1, \ldots, N_n$ be the layer decomposition of Q_n from a vertex u. Then $|N_i| = \binom{n}{i}$ and every vertex in N_i has exactly i neighbours in N_{i-1} and exactly $n - i$ neighbours in N_{i+1} for any $0 \leqslant i \leqslant n$.*

Proof. It is obvious. $\square$

Lemma 3.5 (*Mulder [8], Laborde and Rao Hebbare [7]*). *Let G be a* (0, 2)*-graph. Then G is regular; if n denotes the degree of G, then $|V(G)| \leqslant 2^n$ with equality if and only if G is isomorphic to the hypercube Q_n.*

Proof of Theorem 3.1. We can directly show that any paths and even cycles are distance monotone. By Theorem 2.4 hypercubes with the dimensions at least three are all distance monotone. Hence the sufficiency holds.

We now prove the necessity. Let H be the induced subgraph of G by any interval $I_G(u, v)$. Then H is a distance monotone graph. By Theorem 2.6(3), we know that H is a path if $\delta(H) = 1$, and H is an even cycle if $\delta(H) = 2$. Hence we only show that H is isomorphic to Q_k if $\delta(H) = k \geqslant 3$. We proceed by induction on the minimum degree $k \geqslant 3$.

Suppose that $\delta(H) = k \geqslant 3$. By Lemma 3.3 H is a (0, 2)-graph. By Lemma 3.5, H is k-regular and $|V(H)| \leqslant 2^k$. To prove $H \cong Q_k$, it is sufficient to prove that $|V(H)| \geqslant 2^k$.

Since $I_G(u, v) = I_H(u, v)$, by Lemma 2.2 we have that $D(H) = d_G(u, v) = d_H(u, v)$. Let $N_0, N_1, \ldots, N_D$ be the layer decomposition of H from the vertex u, where $D = D(H)$. Then $N_0 = \{u\}$ and $N_D = \{v\}$.

If $k = 3$, $|N_0| = |N_D| = 1$ and $|N_1| = |N_{D-1}| = 3$. If $D = 2$, u and v have three neighbours in common, contradicting that H is a (0, 2)-graph (Lemma 3.3). So $D \geqslant 3$ and

$$|V(H)| \geqslant |N_0| + |N_1| + |N_{D-1}| + |N_D| \geqslant 8 = 2^3.$$

Hence $H \cong Q_3$.

Suppose that $k \geqslant 4$ and every subgraph of G induced by intervals with the minimum degree $k - 1$ is isomorphic to Q_{k-1}. Since H is k-regular, let $N_1 = \{u_1, u_2, \ldots, u_k\}$, $N_{D-1} = \{v_1, v_2, \ldots, v_k\}$, see Fig. 4. We also have that $D \geqslant 3$ and $N_1 \cap N_{D-1} = \emptyset$.

We claim that $I_G(u_1, v) = I_H(u_1, v)$. Since $I_G(u, v) = I_H(u, v)$ and $u_1 \in N_1$, $d_H(u_1, v) = d_G(u_1, v) = D(H) - 1$. Every u_1, v-geodesic in G is entirely contained in H and thus a shortest path of H since it is extended to u, v-geodesic of G. Hence $I_G(u_1, v) \subseteq I_H(u_1, v)$. For any $w \in I_H(u_1, v)$, $D(H) - 1 = d_G(u_1, v) \leqslant d_G(u_1, w) + d_G(w, v) \leqslant d_H(u_1, w) + d_H(w, v) = D(H) - 1$. Hence the above equalities hold and $w \in I_G(u_1, v)$. So $I_H(u_1, v) \subseteq I_G(u_1, v)$. The claim is verified.

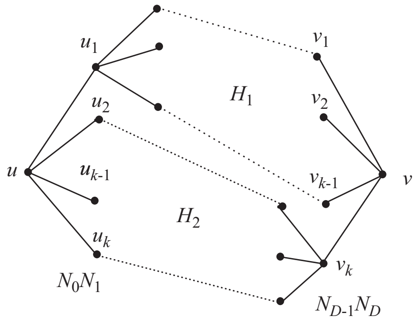

Fig. 4. Illustration for the proof of Theorem 3.1 ($k = 4$).

Let $H_1 := G[I_G(u_1, v)]$. Then $H_1 \subseteq H$. Since H is bipartite (Theorem 2.6(1)), u_1 has exactly $k - 1$ neighbours in N_2, which are all contained in H_1. Hence $d_{H_1}(u_1) = k - 1 \geqslant 3$. Since G is interval distance monotone, H_1 is distance monotone. By Theorem 2.6(3), $\delta(H_1) \geqslant 3$. By Lemmas 3.3 and 3.5, H_1 is $(k-1)$-regular. By the induction hypothesis, $H_1 \cong Q_{k-1}$. Hence $d_{H_1}(u_1, v) = D(H_1) = k - 1 = D(H) - 1$ and $D(H) = k$.

Without loss of generality, assume that $v_k \notin V(H_1)$ and $\{v_1, \ldots, v_{k-1}\} \subseteq V(H_1)$. We take the interval $I_H(u, v_k)$. By the same arguments as above we have $I_G(u, v_k) = I_H(u, v_k)$ and $Q_{k-1} \cong H_2 \subseteq H$, where H_2 denotes the subgraph of G induced by $I_G(u, v_k)$.

We claim that $V(H_1) \cap V(H_2) = \emptyset$. (*)

Let $N_i^1 := N_i \cap V(H_1)$ and $N_i^2 := N_i \cap V(H_2)$ for $i = 0, 1, 2, \ldots, k$. Then $N_0^1 = \emptyset$ and $N_k^2 = \emptyset$. Obviously, any vertex of N_i^1 is at distance $i - 1$ from u_1; and any vertex of N_i^2 is at distance i from u. Since u_1 and v are a pair of diametrical vertices of H_1, and u and v_k are a pair of diametrical vertices of H_2, then $N_1^1, N_2^1, \ldots, N_k^1$ and $N_0^2, N_1^2, \ldots, N_{k-1}^2$ are the layer decompositions of $(k - 1)$-dimensional hypercubes H_1 and H_2 from u_1 and from u, respectively.

We prove that for any $0 \leqslant i \leqslant k$, $N_{k-i}^1 \cap N_{k-i}^2 = \emptyset$ by induction on i. If $i = 0$, $N_k^2 = \emptyset$; if $i = 1$, $N_{k-1}^1 = \{v_1, \ldots, v_{k-1}\}$ and $N_{k-1}^2 = \{v_k\}$. Hence for these two cases, the result holds.

Suppose that for all i with $1 \leqslant i < j \leqslant k-1$, $N_{k-i}^1 \cap N_{k-i}^2 = \emptyset$. Suppose that there exists a vertex $z \in N_{k-j}^1 \cap N_{k-j}^2 \neq \emptyset$. By Lemma 3.4, for H_2, z has exactly $k-j$ neighbours in N_{k-j-1}^2 and $j-1$ neighbours (say $z_1, z_2, \ldots, z_{j-1}$) in $N_{k-(j-1)}^2$; for H_1, z has exactly j neighbours (say $y_1, y_2, \ldots, y_j$) in $N_{k-(j-1)}^1$. By the induction hypothesis, $N_{k-(j-1)}^1 \cap N_{k-(j-1)}^2 = \emptyset$ and $\{y_1, y_2, \ldots, y_j\} \cap \{z_1, z_2, \ldots, z_{j-1}\} = \emptyset$. So

$$d_H(z) \geqslant (k - j) + (j - 1) + j = k + j - 1 > k,$$

contradicting that H is k-regular. Hence $N_{k-j}^1 \cap N_{k-j}^2 = \emptyset$. By the induction we have that $N_i^1 \cap N_i^2 = \emptyset$ for all i and Claim $(*)$ is verified.

By Claim $(*)$ we have that

$$|V(H)| \geqslant |V(H_1)| + |V(H_2)| = 2^{k-1} + 2^{k-1} = 2^k.$$

Hence $H \cong Q_k$ and the necessity holds. □

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